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2007 J. Phys. A: Math. Theor. 40 9097

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# Noncommutativity from spectral flow

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Received 9 May 2007, in final form 1 June 2007

Published 12 July 2007

Online at [stacks.iop.org/JPhysA/40/9097](http://stacks.iop.org/JPhysA/40/9097)

## Abstract

We investigate the transition from second- to first-order systems. Quantum mechanically, this transforms configuration space into phase space and hence introduces noncommutativity in the former. This transition may be described in terms of spectral flow. Gaps in the energy or mass spectrum may become large which effectively truncates the available state space. Using both operator and path integral languages we explicitly discuss examples in quantum mechanics, (light-front) quantum field theory and string theory.

PACS numbers: 11.10.Ef, 11.15.Bt

## 1. Introduction

The last decade has seen a renaissance of the old idea of noncommutative (quantized) spacetime [1], triggered by its reappearance in the context of string and M(atric) theory [2–5]. Applications (and publications) are numerous as is well documented by the recent reports and texts on noncommutative geometry [6, 7], deformation quantization [8], noncommutative field theory [9, 10] and possible phenomenological consequences [11] which include noncommutative approaches to gravity [12, 13], the standard model [14], Lorentz violation [15] and the quantum Hall effect [16, 17].

The latter is based on the quantum mechanics of a particle in a plane pierced by a strong magnetic field. Similar to the string theory scenario it is the presence of the magnetic field that entails noncommutativity between the coordinates in the plane. This physics and its exposition in the papers [18–22] are the original inspiration for the present work. Our main focus is the description of noncommutativity as an *emergent phenomenon* in terms of *spectral flow*.

In particular, we analyse how commutative spaces become noncommutative in special limits of quantum mechanical theories. The limits to be studied appear initially to be unrelated. However, we will unveil that there are features common to all, and indeed that there is a unifying picture.

Let us outline our approach, using a generic (spectral flow) parameter  $\lambda$  to characterize the limits in question as  $\lambda \rightarrow 0$ . From an action principle point of view, the limits we consider correspond to terms quadratic in time derivatives (velocities) vanishing or being rendered negligible compared to other terms. As  $\lambda \rightarrow 0$  the remaining terms are at most linear in time derivatives, so that we move from a second to a first-order system. This implies a significant alteration to the theory, as the definition of the conjugate momenta and therefore the Poisson brackets of the theory will be quite different to when  $\lambda \neq 0$ .

When we quantize, Poisson brackets are replaced by commutators of operators. In an operator picture we observe that the energy spectrum is  $\lambda$ -dependent. As we take the limit some portion of states becomes highly excited and decouples from the theory—effectively this spectral flow truncates the available state space. Operators which commute at  $\lambda \neq 0$  do so because of cancellations between the various modes of the operators (think of working with the Fourier modes of a scalar field). As some of these modes are decoupled at  $\lambda = 0$  such cancellations are incomplete and operators may fail to commute. Typically, it is the configuration space (spacetime coordinate or field configuration) operators which become noncommutative in the limit, as the spectral flow takes us to a theory where configuration space becomes phase space.

We will also investigate the limit from a functional perspective, where we show that functionals such as the vacuum and transition amplitudes have natural interpretations as  $\lambda \rightarrow 0$  in terms of functionals in the first-order theory. Here one must carefully take into account changes to the true degrees of freedom (the arguments of functionals) which occur because of the shift from configuration to phase space.

This paper is organized as follows. In section 2 we discuss the quantum mechanics of a particle in a magnetic field. In the limit in which the magnetic field is large compared to the particle mass we observe that coordinates become noncommutative. We describe this limit in the operator language as a projection onto the lowest energy level. From a functional perspective we show how to take account of the change from configuration space to phase space, giving explicit examples of the first-order limit of second-order transition amplitudes. We conclude this section with a discussion of an analogous situation in string theory, where the presence of a strong ‘magnetic’ field leads to an effective lower dimensional noncommutative field theory.

In section 3 we study the nonrelativistic limit of a quantum field theory, in which states of high energy and momentum are decoupled. We see that the second-order Klein–Gordon equation becomes the first-order Schrödinger equation and show that only particle-number conserving interactions survive the nonrelativistic limit.

In section 4 we describe light-front quantum field theory as a limiting transition to quantizing on null-planes. Using ‘almost’ light-front coordinates we describe the energy spectrum and show that half of the mass shell energies are decoupled in the light-front limit. We then show explicitly how this leads to non-zero commutators of the field with itself, and describe the vacuum functional and time evolution generator in the light-front limit.

We present our conclusion in section 5. The appendices contain some review material on relevant functional integrals.

## 2. Particle in a strong magnetic field

### 2.1. Operator approach

Consider a nonrelativistic particle moving in the  $xy$  plane under the influence of a constant magnetic field of magnitude  $B$  in the  $z$ -direction. Upon quantization this is the problem

originally solved by Landau in 1930 [23]. The system is described by the standard Lagrangian [24]

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + B\dot{x}y. \tag{1}$$

The conjugate momenta are

$$p_x = m\dot{x} + By \equiv \pi_x + By, \quad p_y = m\dot{y} \equiv \pi_y, \quad p_z = m\dot{z} \equiv \pi_z, \tag{2}$$

and obey the equations of motion

$$\dot{p}_x = 0, \quad \dot{p}_y = B\dot{x}, \quad \dot{p}_z = 0. \tag{3}$$

These imply three conserved quantities  $p_z$ ,  $x_0$  and  $y_0$  with the latter two given by

$$y_0 = y + \frac{\dot{x}}{\omega}, \quad x_0 = x - \frac{\dot{y}}{\omega}, \tag{4}$$

where  $\omega = B/m$  is the usual cyclotron frequency. The relevance of these two operators in the quantum theory was first noted by Johnson and Lippmann [25].

Upon performing a Legendre transformation the Hamiltonian is found to be

$$H = p_x\dot{x} + p_y\dot{y} - L = \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(y - y_0)^2 + \frac{p_z^2}{2m}. \tag{5}$$

We introduce ladder operators

$$a = \left(\frac{m\omega}{2}\right)^{1/2} \left(y - y_0 + \frac{i}{m\omega}p_y\right) \tag{6}$$

$$a^\dagger = \left(\frac{m\omega}{2}\right)^{1/2} \left(y - y_0 - \frac{i}{m\omega}p_y\right), \tag{7}$$

in terms of which the Hamiltonian may be written as

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{p_z^2}{2m}. \tag{8}$$

Obviously this represents a harmonic oscillator in  $y$  shifted by  $y_0$ , as in (4), and free motion in  $z$ .

As operators, the conserved quantities (4) commute with the Hamiltonian and the kinematical (not conjugate) momenta. Their commutators with the coordinate operators are

$$[\hat{x}_0, \hat{y}_0] = iB^{-1}, \tag{9}$$

$$[\hat{x}_0, \hat{y}] = [\hat{x}, \hat{y}_0] = iB^{-1} \tag{10}$$

$$[\hat{x}_0, \hat{x}] = [\hat{y}_0, \hat{y}] = 0. \tag{11}$$

Eigenstates of the Hamiltonian are labelled by the oscillator (Landau) level  $n$  and  $p_z$  but are infinitely degenerate with respect to  $p_x$ ,

$$\hat{H}|n, p_x, p_z\rangle = E_n(p_z)|n, p_x, p_z\rangle, \tag{12}$$

as the energy is independent of  $p_x$ ,

$$E_n(p_z) \equiv \omega\left(n + \frac{1}{2}\right) + \frac{p_z^2}{2m}. \tag{13}$$

Note that the level spacing  $\omega = B/m$  becomes large for  $B \gg m$ . In this case one expects that transitions between Landau levels are strongly suppressed and that any dynamics will be restricted to the lowest level,  $n = 0$  [18, 20, 22]. The projection onto the latter is given by the operator

$$\mathbb{P} = \int \frac{dp_z dp_x}{(2\pi)^2} |0, p_x, p_z\rangle \langle 0, p_x, p_z|. \quad (14)$$

In what follows we evaluate the projected commutator  $[\mathbb{P}\hat{x}\mathbb{P}, \mathbb{P}\hat{y}\mathbb{P}]$  and show that it is nonvanishing. As a preparation we note

$$\mathbb{P}\hat{y}\mathbb{P} = \mathbb{P} \left[ \hat{y}_0 + \frac{1}{\sqrt{2m\omega}} (\hat{a} + \hat{a}^\dagger) \right] \mathbb{P} = \mathbb{P}\hat{y}_0\mathbb{P}. \quad (15)$$

The second equality follows from  $\hat{a}\mathbb{P} = 0$  since  $\mathbb{P} \sim |0, p_x, p_z\rangle$  and  $\mathbb{P}a^\dagger\mathbb{P} = 0$  since  $\mathbb{P}a^\dagger\mathbb{P} \sim \langle p'_z, p'_x, 0 | 1, p_x, p_z\rangle = 0$ . Similarly,

$$\mathbb{P}\hat{x}\mathbb{P} = \mathbb{P} \left[ \hat{x}_0 + \frac{1}{i\sqrt{2m\omega}} (\hat{a} - \hat{a}^\dagger) \right] \mathbb{P} = \mathbb{P}\hat{x}_0\mathbb{P}. \quad (16)$$

Now,  $\hat{y}_0$  commutes with the Hamiltonian,  $\hat{p}_x$  and  $\hat{p}_z$ , so

$$\mathbb{P}\hat{y}_0\mathbb{P} = \mathbb{P}\hat{y}_0 = \hat{y}_0\mathbb{P}, \quad (17)$$

using  $\mathbb{P}^2 = \mathbb{P}$ . So, finally,

$$[\mathbb{P}\hat{x}\mathbb{P}, \mathbb{P}\hat{y}\mathbb{P}] = [\mathbb{P}\hat{x}_0\mathbb{P}, \mathbb{P}\hat{y}_0\mathbb{P}] = \mathbb{P}[\hat{x}_0, \hat{y}_0]\mathbb{P} = iB^{-1}\mathbb{P}. \quad (18)$$

We see that the projection onto the lowest energy level results in a non-zero commutator between the (projected) position operators.

The commutator may be explained in a simple fashion [18, 20, 22] by performing the limit  $m/B \rightarrow 0$  (small mass/large field) in the Lagrangian (1). To retain a nontrivial theory we add an arbitrary potential  $V(x, y)$  and arrive at the *first-order* Lagrangian

$$L = B\dot{x}y - V(x, y). \quad (19)$$

The Poisson bracket or commutator is read off from the first term (the ‘canonical one-form’ [26]) which yields

$$[x, y] = iB^{-1}. \quad (20)$$

Hence, the configuration space variables  $x$  and  $y$  become a canonical pair and thus define a phase space on which one has the Hamiltonian

$$h(x, y) \equiv V(x, y), \quad (21)$$

with  $B$  playing the role of the momentum conjugate to  $x$  (Peierls’ substitution [27–29]). The Hilbert space of states may be taken to be  $\mathbb{L}^2(\mathbb{R})$  consisting of wavefunctions  $\Psi = \Psi(x)$ . The emerging picture will be the basis of the following subsection.

## 2.2. Path integral approach

We have seen that at a classical level a first-order theory is obtained simply by deleting the kinetic term in the Lagrangian of the second-order theory, though quantum mechanically the limit is somewhat more subtle. In this subsection we will study the limit from a functional viewpoint.

We begin with a typical wavefunction in the second-order theory, say the amplitude describing particle transition from  $(x_i, y_i)$  to  $(x_f, y_f)$  in time  $t = T$ . In the Euclidean path

integral language this amplitude is the sum over all paths between the two points weighted with the exponent of the classical (Euclidean) action,

$$\langle x_f, y_f | e^{-\hat{H}T} | x_i, y_i \rangle = \int \mathcal{D}y \int \mathcal{D}x \exp - \int_0^T dt L \Big|_{x(0)=x_i, y(0)=y_i}^{x(T)=x_f, y(T)=y_f}, \quad (22)$$

with  $L$  the Euclidean version of (1) together with a potential  $V(x, y)$ . For a review of this and similar constructions, see appendices A and B. For clarity we will neglect any  $z$  dependence. Let  $\hat{H}$  and  $E_0$  ( $\hat{h}$  and  $\epsilon_0$ ) be the Hamiltonian and vacuum energy in the second (first) order theory. In this section we will discuss the following operation and show that it gives first-order transition amplitudes in the limit of small mass/large  $B$ ,

$$\lim_{m \rightarrow 0} \int dy_f \int dx_i \langle x_f, y_f | e^{-(\hat{H}-E_0)T} | x_i, y_i \rangle = \langle x_f | e^{-(\hat{h}-\epsilon_0)T} | x_i \rangle. \quad (23)$$

There are two points to consider. The first is the subtraction of the vacuum energy and the second the integration over *half* of the boundary degrees of freedom in the second-order theory. In the operator formalism it has been seen that the energy spectrum undergoes a flow which decouples excited states as  $m/B \rightarrow 0$ . In the case of a particle in a magnetic field this constrains the particle to lie in the ground state, that is the lowest Landau level. Strictly, however, even the ground state of this system acquires a divergent energy, ( $\omega/2 = B/2m \rightarrow \infty$  in the case of  $V = 0$ ), and this should be subtracted from the Hamiltonian in order to arrive at a meaningful system. This is most clearly seen using the spectral decomposition of the transition amplitude,

$$\langle x_f, y_f | e^{-\hat{H}T} | x_i, y_i \rangle = \sum_{n \geq 0, d} \psi_{n,d}^*(x_f, y_f) e^{-E_n T} \psi_{n,d}(x_i, y_i), \quad (24)$$

where the sum (which represents any combination of discrete and continuous measure) is over all energy eigenvalues  $E_n$  of the Hamiltonian and degeneracies  $d \equiv (p_x, p_z)$  of those energy levels [22]. If the eigenvalues flow to infinity as the mass decreases we see that all terms in this series are exponentially damped and in the massless limit this expression is null. If, however, we subtract the vacuum energy from the Hamiltonian then this sum becomes

$$\begin{aligned} \langle x_f, y_f | e^{-(\hat{H}-E_0)T} | x_i, y_i \rangle &= \sum_d \psi_{0,d}^*(x_f, y_f) \psi_{0,d}(x_i, y_i) \\ &+ \sum_{n \geq 1, d} \psi_{n,d}^*(x_f, y_f) e^{-\Delta E_n T} \psi_{n,d}(x_i, y_i), \end{aligned} \quad (25)$$

where  $\Delta E_n \equiv E_n - E_0$ . The first term now survives the limit, the caveat being that these manipulations are only well defined in Euclidean space.

Moving on to the second point we consider the integration over half of the degrees of freedom on the ‘boundary’  $t = 0$  and  $t = T$ . This has a natural interpretation in the second-order theory—rather than consider the amplitude for a transition between points  $(x_i, y_i)$  and  $(x_f, y_f)$  we instead ask for the amplitude for transition between points  $x_i$  and  $x_f$  for any initial and final values of  $y$ . Now, as already stated, in the massless limit the operators  $\hat{x}$  and  $\hat{y}$  form a conjugate pair. In taking the limit from the second-order theory, where these variables are independent, we must choose which of  $x$  and  $y$  to consider as a coordinate and which a momentum. In the first-order theory our prescription corresponds to a particular choice of polarization or (Schrödinger) representation, namely that where we diagonalize the operator  $\hat{x}$ .

In general we must choose one linear combination of  $\hat{x}$  and  $\hat{y}$  to be diagonalized and compute the transition amplitude between eigenstates of these operators. The remaining degrees of freedom should be integrated over at the boundary. In this way the configuration

space path integral becomes a phase space path integral in the first-order theory. If we choose to represent states in the massless limit by wavefunctions  $\Psi(x)$ , then we integrate over all possible values of the momentum  $y_f$  and  $y_i$  at the boundaries on the left-hand side of (23) (or the right-hand side of (22)) to arrive at

$$\langle x_f | e^{-\hat{h}T} | x_i \rangle = \int \mathcal{D}y \int \mathcal{D}x \exp \int_0^T [iBx\dot{y} - h(x, y)] \Big|_{x(0)=x_i}^{x(T)=x_f}, \tag{26}$$

which, upon changing variables  $p = By$ , is a standard phase space path integral describing a transition amplitude in the lower dimensional first-order theory. We will give explicit examples below.

2.3. Example one: no external potential

A simple example of the above is given by the particle in a magnetic field with no external potential, for which the transition amplitudes in the first-order theory are simply

$$\langle x_f | e^{-\hat{h}T} | x_i \rangle |_{h=0} = \delta(x_f - x_i). \tag{27}$$

We would like to relate this amplitude to the ‘projected’ transition amplitude in the  $m \neq 0$  theory,

$$\langle x_f, y_f | \mathbb{P} e^{-(\hat{H}-E_0)T} | x_i, y_i \rangle. \tag{28}$$

We label energy eigenstates by the oscillator level  $n$  and an eigenvalue of  $\hat{p}_x = B\hat{y}_0$ . The projection operator is then written as an integral over the degenerate ground states  $|0, p_x\rangle$ ,

$$\mathbb{P} = \int \frac{dp_x}{2\pi} |0, p_x\rangle \langle p_x, 0|, \tag{29}$$

normalized such that  $\langle 0, p_x | q_x, 0 \rangle = 2\pi \delta(p_x - q_x)$  and so  $\mathbb{P}^2 = \mathbb{P}$ . The projected amplitude (28) may then be written

$$\langle x_f, y_f | \mathbb{P} e^{-(\hat{H}-E_0)T} | x_i, y_i \rangle = \int \frac{dp_x}{2\pi} \langle x_f, y_f | 0, p_x \rangle \langle 0, p_x | x_i, y_i \rangle \tag{30}$$

$$= \int \frac{dp_x}{2\pi} \psi_{0,p_x}(x_f, y_f) \psi_{0,p_x}^*(x_i, y_i). \tag{31}$$

Comparing this with the spectral decomposition of the full transition amplitude (25) we see that the two coincide at large times (if we rotate to Euclidean space). The effect of the projection is to restrict intermediate states of the system to the ground state, so that transition amplitudes do not obtain contributions from excited states.

To calculate the projected amplitude we first derive the explicit form of the ground-state wavefunction. In the above representation this is given by the two conditions

$$\begin{aligned} \hat{p}_x \psi_{0,p}(x, y) &\equiv -i\partial_x \psi_{0,p}(x, y) = p\psi_{0,p}(x, y), \\ \hat{a} \psi_{0,p}(x, y) &\equiv \left(\frac{B}{2m}\right)^{1/2} \left(y - \frac{p}{B} + \frac{1}{B}\partial_y\right) \psi_{0,p}(x, y) = 0, \end{aligned} \tag{32}$$

which are respectively plane wave and harmonic oscillator equations. They have the solutions

$$\psi_{0,p}(x, y) = N_p \exp \left\{ ixp - \frac{B}{2}(y - pB^{-1})^2 \right\}, \quad N_p^2 = \sqrt{B/\pi}. \tag{33}$$

Using these functions we follow the procedure of the previous subsection, integrating (30) over  $y_f$  and  $y_i$ , cf (23),

$$\int dy_f dy_i \langle x_f, y_f | \mathbb{P} e^{-(\hat{H}-E_0)T} | x_i, y_i \rangle = 2\sqrt{\frac{\pi}{B}} \delta(x_f - x_i), \tag{34}$$

recovering, up to an irrelevant normalization effect, the trivial transition amplitude (27) of the first-order theory.

2.4. Example two: harmonic oscillator potential

We now give a non-trivial example of the massless limit. Consider the (Euclidean) action,

$$S_{\mathcal{E}}(m) = \int_0^T dt \frac{m}{2} (\dot{x}^2 + \dot{y}^2) - iB\dot{x}y + \frac{\lambda^2}{2}(x^2 + y^2). \tag{35}$$

Setting  $m = 0$  we arrive at the Euclidean action of the harmonic oscillator (upon changing variables  $p = By$ ) with frequency  $\mu = \lambda/B$  and mass  $B^2/\lambda$ . The phase space integral which quantizes the  $m = 0$  action is

$$\int \mathcal{D}x \mathcal{D}p \exp -S_{\mathcal{E}}(0) \Big|_{x(0)=x_i}^{x(T)=x_f}, \tag{36}$$

where the momentum  $p = By$  has a free boundary. This result is well known and in this section we will recover it from the massless limit of the quantized second-order theory.

For  $m \neq 0$ , following (23), we integrate over boundary values of  $y$  in the transition amplitude,

$$\int dy_i dy_f \langle x_f, y_f | e^{-i\hat{H}T} | x_i, y_i \rangle \equiv \int dy_i dy_f \int \mathcal{D}y \int \mathcal{D}x \exp -S_{\mathcal{E}}(m) \Big|_{x(0)=x_i, y(0)=y_i}^{x(T)=x_f, y(T)=y_f}. \tag{37}$$

The integrals are Gaussian and easily computed, the result is

$$\frac{1}{\det^{1/2}} \exp \left[ a_0(m) + \sum_{n=1}^{\infty} a_n(m) \right], \tag{38}$$

where the determinant factor is given by

$$\det \equiv \prod_{n=1}^{\infty} \mathcal{K}_n(m) \quad \text{with} \quad \mathcal{K}_n(m) = \left( \lambda + \frac{mn^2\pi^2}{T^2} \right)^2 + \frac{B^2n^2\pi^2}{T^2}, \tag{39}$$

and the exponentiated terms are

$$a_0(m) = -\frac{T\lambda}{6} (x_f^2 + x_i^2 + x_f x_i) - \frac{m}{2T} (x_f - x_i)^2 - \frac{B^2}{2\lambda T} (x_f - x_i)^2, \tag{40}$$

$$a_n(m) = \frac{\lambda^2 T}{n^2 \pi^2} ((-)^n x_f - x_i)^2 \left( \lambda + \frac{mn^2\pi^2}{T^2} \right) \mathcal{K}_n(m)^{-1}. \tag{41}$$

The sum over  $a_n(m)$  behaves as  $n^{-4}$  for  $n$  large and is convergent, although the result is an unenlightening combination of hypergeometric functions which nevertheless gives the



expected result as  $m \rightarrow 0$ . Rather than detail this we illustrate it by performing the (in this case) equivalent operations of taking  $m \rightarrow 0$  and then performing the sum,

$$\sum_{n=1}^{\infty} a_n(0) = (x_f^2 + x_i^2) \left[ \frac{\lambda T}{6} + \frac{B^2}{2\lambda T} - \frac{B \cosh(\mu T)}{2 \sinh(\mu T)} \right] - x_f x_i \left[ \frac{B^2}{\lambda T} - \frac{\lambda T}{6} - \frac{B}{\sinh(\mu T)} \right]. \tag{42}$$

Therefore, in the massless limit we find the exponential of

$$a_0(0) + \lim_{m \rightarrow 0} \sum a_n(m) = -\frac{B}{2 \sinh(\mu T)} (x_f^2 \cosh(\mu T) + x_i^2 \cosh(\mu T) - 2x_f x_i), \tag{43}$$

which is the classical Euclidean action of the harmonic oscillator. We now turn to the determinant factor multiplying this exponential, which is divergent and must be regulated *before* we can take  $m \rightarrow 0$ . Zeta-function regularization gives

$$\det \rightarrow \cosh \left( \frac{BT}{m} + 2\mu T + \mathcal{O}(m) \right) - \cosh \left( \frac{BT}{m} \right). \tag{44}$$

up to an  $m$  and  $T$ -independent constant prefactor. The  $m \rightarrow 0$  limit of the determinant does not exist as there is an essential singularity at  $m = 0$ . However, as stated in (23), we should subtract the vacuum energy  $E_0$  from the Hamiltonian before taking the limit. This subtraction pre-multiplies the transition amplitude by  $\exp(E_0 T)$ . The vacuum energy for this system is given in [18],

$$E_0 = \sqrt{\frac{B^2}{4m^2} + \frac{\lambda}{m}} = \frac{B}{2m} + \mu + \mathcal{O}(m). \tag{45}$$

The  $m \rightarrow 0$  limit of the product of the determinant factor and  $\exp(E_0 T)$  exists (at least for  $m \rightarrow 0^+$ ),

$$\lim_{m \rightarrow 0} \frac{e^{E_0 T}}{\det^{1/2}} = e^{\mu T/2} (\sinh(\mu T))^{-1/2}. \tag{46}$$

In total we therefore find

$$\begin{aligned} \lim_{m \rightarrow 0} \int dy_f \int dy_i \langle x_f, y_f | e^{-(\hat{H}-E_0)T} | x_i, y_i \rangle &= e^{\mu T/2} (\sinh(\mu T))^{-1/2} \\ &\times \exp -\frac{B}{2 \sinh(\mu T)} (x_f^2 \cosh(\mu T) + x_i^2 \cosh(\mu T) - 2x_f x_i), \end{aligned} \tag{47}$$

which is the transition amplitude for the harmonic oscillator, calculated with the vacuum energy  $\mu/2$  subtracted from the Hamiltonian. This provides a non-trivial example of the prescription (23) describing the transition between a second- and first-order theory.

### 2.5. A stringy analogue

The particle in a strong magnetic field has a well-known counterpart system in string theory (see [30] and references therein). We consider neutral open strings with Dp-branes on which the strings end. The worldsheet action is

$$\begin{aligned} S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d\sigma d\tau \partial_{\tau} X^i g_{ij}(X) \partial_{\tau} X^j - \partial_{\sigma} X^i g_{ij}(X) \partial_{\sigma} X^j \\ - \int d\tau \partial_{\tau} X^{\mu} B_{\mu\nu} X^{\nu} \Big|_{\sigma=\pi} + \int d\tau \partial_{\tau} X^{\mu} B_{\mu\nu} X^{\nu} \Big|_{\sigma=0}. \end{aligned} \tag{48}$$

Here  $g_{\mu\nu}(X)$  and  $B_{\mu\nu}(X)$  describe the geometry of target space. The worldsheet  $\Sigma$  for free strings, parameterized by  $\tau$  and  $\sigma$ , is an infinite strip in the  $\tau$  direction with width  $\pi$  in  $\sigma$ . We

will consider a flat target space,  $g_{\mu\nu}(X) = \eta_{\mu\nu}$ , and take the two-form flux  $B_{\mu\nu}$ , representing a magnetic field on the brane, to be constant.

The equation of motion and boundary conditions in the  $p + 1$  directions parallel to the brane are

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0, \quad \partial_\sigma X^\mu + 2\pi\alpha' \partial_\tau X^\nu B_\nu{}^\mu = 0 \quad \text{at } \sigma = 0, \pi, \tag{49}$$

which have the solution [31]

$$X^\mu = x^\mu + p^\mu \tau - 2\pi\alpha' p^\nu B_\nu{}^\mu \sigma + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (i a_n^\mu \cos n\sigma - 2\pi\alpha' a_n^\nu B_\nu{}^\mu \sin n\sigma). \tag{50}$$

The commutation relations are

$$\begin{aligned} [x^\mu, x^\nu] &= (2\pi\alpha')^2 i (M^{-1} B)^{\mu\nu}, \\ [x^\mu, p^\nu] &= 2\alpha' i (M^{-1})^{\mu\nu}, \\ [a_n^\mu, a_m^\nu] &= 2\alpha' (M^{-1})^{\mu\nu} n \delta_{n+m}, \end{aligned} \tag{51}$$

with all others vanishing and where  $M_{ij} = g_{ij} - (2\pi\alpha')^2 B_{ik} g^{kr} B_{rj}$ .

There is a low energy limit of this theory which parallels that of the particle case and which we will discuss shortly. First note that there is a noncommutativity inherent in this system before we take any limit. It is straightforward to check using (51) that

$$[X^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] = 0, \tag{52}$$

for all  $\sigma, \sigma' \in (0, \pi)$  and whenever  $\sigma$  and  $\sigma'$  are not both 0 or  $\pi$ . This means that in the bulk of target space, away from the branes, spacetime is described by commutative coordinates. However, the commutators between endpoints of the string are non-zero,

$$\begin{aligned} [X^\mu(\pi, \tau), X^\nu(\pi, \tau)] &= -(2\pi\alpha')^2 i (M^{-1} B)^{\mu\nu}, \\ [X^\mu(0, \tau), X^\nu(0, \tau)] &= (2\pi\alpha')^2 i (M^{-1} B)^{\mu\nu}. \end{aligned} \tag{53}$$

The ends of the string therefore describe noncommutative coordinates on the branes (closed strings are insensitive to this effect and see spacetime as a commutative manifold). Let us now tie this in our to earlier discussions. A low energy limit of this theory [5] may be taken in which the string coupling and metric scale as

$$\alpha' = \sqrt{\epsilon} \rightarrow 0, \quad g_{\mu\nu} \sim \epsilon \rightarrow 0, \tag{54}$$

with the magnetic field  $B_{ij}$  held fixed—so this is a limit in which  $B$  is strong compared to other fields, as in the particle case. In studying this limit it is sufficient to focus on a pair of coordinates so that all metrics and fields become two-by-two matrices,

$$g_{\mu\nu} = \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_{\mu\nu} = B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (M^{-1})^{\mu\nu} = \frac{1}{\epsilon + 4\pi^2 B^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{55}$$

In this limit, the commutators (51) behave as

$$[x^\mu, x^\nu] = i \frac{4\pi^2 B}{\epsilon + 4\pi^2 B^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow -i (B^{-1})^{\mu\nu}, \tag{56}$$

$$[x^\mu, p^\nu] \sim i\sqrt{\epsilon} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [a_n^\mu, a_m^\nu] \sim \sqrt{\epsilon} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{57}$$

so that in this limit the theory undergoes a spectral flow in which higher energy states of the open string, created by the action of the oscillators  $a_{-n}^\mu$ , are suppressed by the vanishing of  $\alpha'$  (which corresponds to the field theory limit). The scaling of the metric also decouples

closed string states from the theory. We are left with only the lowest energy states in which the degrees of freedom are the end points of the open string. In this limit the endpoint commutation relations (53) become

$$\begin{aligned} [X^\mu(\pi, \tau), X^\nu(\pi, \tau)] &= i \begin{pmatrix} 0 & -B^{-1} \\ B^{-1} & 0 \end{pmatrix} = i(B^{-1})^{\mu\nu}, \\ [X^\mu(0, \tau), X^\nu(0, \tau)] &= -i \begin{pmatrix} 0 & -B^{-1} \\ B^{-1} & 0 \end{pmatrix} = -i(B^{-1})^{\mu\nu}, \end{aligned} \quad (58)$$

in analogy to the quantum mechanical commutator (20). Under (54) the first terms of the action (48), second order in derivatives, are suppressed and it is only the boundary terms which survive the limit,

$$S \rightarrow - \int d\tau \partial_\tau X^\mu B_{\mu\nu} X^\nu \Big|_{\sigma=0}^{\sigma=\pi} + \int d\tau \partial_\tau X^\mu B_{\mu\nu} X^\nu \Big|_{\sigma=0}, \quad (59)$$

again in direct analogy to the particle action (19). From this action we may immediately recover (58).

This low energy limit therefore describes a spectral flow in which higher excitations of the string are suppressed, and from the action we see that this corresponds to a transition from a second- to first-order theory, where the degrees of freedom are noncommutative particle-like coordinates on the branes. It is in this way that non-commutative field theories arise on the branes as the low energy limit of string theories [32].

### 3. The nonrelativistic limit

Throughout the remainder of this paper spacetime is  $D = 1 + d$  dimensional unless otherwise stated. Functional integrals over time dependent fields will be written  $\mathcal{D}\phi$  while integrals over configurations at constant time will be written  $\mathcal{D}\varphi$ .

#### 3.1. Free theory

Consider the action of a free relativistic scalar particle given by the bilinear expression

$$S = \int d^D x \phi^\dagger (-\square - m^2) \phi. \quad (60)$$

This action describes either a complex scalar field  $\phi$  or a real scalar field  $\phi + \phi^\dagger$  (where, for example,  $\phi$  may be taken to be the positive frequency part of the real field).

In the nonrelativistic limit, all energies and momenta are small compared to the particle mass  $m$ . Following chapter III.5 of [33] we define a new field  $\Phi$  such that

$$\phi(x) = \frac{e^{-imt}}{\sqrt{2m}} \Phi(x) \quad \Rightarrow \quad \phi^\dagger(x) = \frac{e^{imt}}{\sqrt{2m}} \Phi^\dagger(x). \quad (61)$$

The motivation for this is that a mode carrying kinetic energy  $E$ , oscillates as  $\exp -i(E + m)t$ . When the kinetic energy is small compared to  $m$  the definitions (61) factor out the rapid oscillations which are not admissible in a nonrelativistic approximation.

This argument can only hold if we restrict the energy and momentum of the field, so to proceed we work in momentum space. Defining the Fourier transforms of fields and their conjugates by

$$\phi(x) = \int \frac{d^D p}{(2\pi)^D} e^{-ip \cdot x} \tilde{\phi}(p), \quad \phi^\dagger(x) = \int \frac{d^D p}{(2\pi)^D} e^{ip \cdot x} \tilde{\phi}^\dagger(p), \quad (62)$$

the action becomes

$$S = \int \frac{d^D p}{(2\pi)^D} \tilde{\phi}^\dagger(p)(p^2 - m^2)\tilde{\phi}(p), \tag{63}$$

and one may verify the relations

$$\tilde{\phi}(p) = \frac{1}{\sqrt{2m}} \tilde{\Phi}(p_0 - m, \mathbf{p}), \quad \tilde{\phi}^\dagger(p) = \frac{1}{\sqrt{2m}} \tilde{\Phi}^\dagger(p_0 - m, \mathbf{p}). \tag{64}$$

We now assume that all energies and momenta are small compared to the particle mass  $m$ ,

$$E \equiv p_0 - m \ll m, \quad |\mathbf{p}| \ll m, \tag{65}$$

from which we obtain the nonrelativistic approximation

$$\begin{aligned} p^2 - m^2 &= (E + m - E_p)(E + m + E_p) \\ &\simeq (E - \mathbf{p}^2/2m)(E + 2m + \mathbf{p}^2/2m) \\ &\simeq (E - \mathbf{p}^2/2m)2m. \end{aligned} \tag{66}$$

We wish to insert this approximation into the action (63). However, it is clear that this is only consistent if we introduce a large momentum cutoff. Clearly, this is quite natural in the relativistic theory where we have to regulate ultra-violet divergences anyhow. Hence, we change variables  $E = p_0 - m$  in (63) and impose cutoffs in both  $E$  and momentum  $\mathbf{p}$ . For clarity we refer to only a single cutoff  $\Lambda$  with  $\Lambda \ll m$ . We may now insert our approximation (66) into the action,

$$S \rightarrow \int^\Lambda \frac{dE}{2\pi} \frac{d^d p}{(2\pi)^d} \tilde{\Phi}^\dagger(E, \mathbf{p}) \left( E - \frac{\mathbf{p}^2}{2m} \right) \tilde{\Phi}(E, \mathbf{p}). \tag{67}$$

Inverting the Fourier transform we arrive at the nonrelativistic action in coordinate space,

$$S \rightarrow \int d^D x d^D y \Phi^\dagger(y) \delta_\Lambda(x - y) \left( i\partial_t + \frac{\Delta}{2m} \right) \Phi(x), \tag{68}$$

where short distance divergences are controlled by the regulated delta function

$$\delta_\Lambda(x - y) = \int^\Lambda \frac{d^D p}{(2\pi)^D} \exp -ip \cdot (x - y). \tag{69}$$

As we take the cutoff to infinity we arrive at the nonrelativistic action [34]

$$S_{\text{NR}} = \int d^D x \Phi^\dagger(x) \left( i\partial_t + \frac{\Delta}{2m} \right) \Phi(x). \tag{70}$$

Unlike (60) this is now linear in the time derivative  $\partial_t \Phi$  and hence the nonrelativistic Schrödinger equation,

$$(i\partial_t + \Delta/2m)\Phi(\mathbf{x}, t) = 0, \tag{71}$$

correctly becomes first order in the time derivative. As a result (and completely analogous to the Dirac field) the Schrödinger matter field does not commute with its conjugate. Rather we read off the commutator from the  $\Phi^\dagger i\partial_t \Phi$  term, given by  $i$  times the Poisson bracket,

$$[\Phi(t, \mathbf{x}), \Phi^\dagger(t, \mathbf{y})] = \delta(\mathbf{x} - \mathbf{y}). \tag{72}$$

Both (70) and (72) coincide with the expressions derived in a slightly different way in the recent text [33].

### 3.2. Particle number

The Schrödinger matter field still describes an (albeit nonrelativistic) many-body theory. However, the different particle number sectors are separated by huge gaps of order  $m$  so that particle number becomes conserved. Note that this is also true for nonrelativistic bound states which have binding energies  $E_B$  satisfying

$$E_B \ll |\mathbf{p}| \ll M, \quad (73)$$

where  $\mathbf{p}$  is a typical constituent momentum and  $M$  the bound state mass. Hence, say for a two-particle bound state, we have  $M = 2m - E_B \simeq 2m$ , so that  $M$  is close to the 2-particle threshold and hence separated from the one-particle mass-shell by a gap of almost  $m$ .

The suppression of number changing interactions in the nonrelativistic limit may be seen from an action principle. Reality requires that polynomial interaction terms take one of the forms

$$S_1 = \lambda_1 \int d^D x \phi^{\dagger n}(x) \phi^n(x) \quad (74)$$

$$S_2 = \lambda_2 \int d^D x \phi^{\dagger r}(x) \phi^n(x) + \text{h.c.}, \quad r \neq n. \quad (75)$$

Changing to the new fields of (61), and performing a change of variables  $E = p_0 - m$  for each energy integration variable we have

$$S_1 = \lambda_1 \int^{\Lambda} \tilde{\Phi}^{\dagger}(E_1) \dots \tilde{\Phi}^{\dagger}(E_n) \tilde{\Phi}(E_{n+1}) \dots \tilde{\Phi}(E_{2n}) \\ \times \delta(E_1 + \dots + E_n - E_{n+1} - \dots - E_{2n}), \quad (76)$$

$$S_2 = \lambda_2 \int^{\Lambda} \tilde{\Phi}^{\dagger}(E_1) \dots \tilde{\Phi}^{\dagger}(E_r) \tilde{\Phi}(E_1) \dots \tilde{\Phi}(E_n) \\ \times \delta(E_1 + \dots + E_r - E_1 - \dots - E_n + (r - n)m) + \text{h.c.}, \quad (77)$$

where the integration is over all  $E_i$  and  $\mathbf{p}_i$ . We have suppressed the momentum dependence for clarity, and to make a consistent relativistic approximation we again understand all integrals to be ultra-violet regulated by  $\Lambda$ .

In the nonrelativistic approximation, when all energies are small compared to  $m$ , we see that the delta function in  $S_2$  loses support because of the non-zero multiple of  $m$ . These are precisely the interactions which do not conserve particle number, or alternatively, which do not conserve nonrelativistic energy. Hence, only actions of the form  $S_1$ , which conserve both particle number and nonrelativistic energy, survive the nonrelativistic limit.

### 3.3. Projection and particle number

We may now ask, for example, how to get from the field  $\Phi$  to the one-particle sector and the associated Schrödinger wavefunction  $\Psi(x)$ ? The answer is well known: if  $|0\rangle$  and  $|\mathbf{p}\rangle$  denote the vacuum and a one-particle state of momentum  $\mathbf{p}$  then, at  $t = 0$ ,

$$\Psi_{\mathbf{p}}(x) = \langle 0 | \Phi(\mathbf{x}) | \mathbf{p} \rangle = e^{i\mathbf{p} \cdot \mathbf{x}} \quad (78)$$

is the plane wave solution of the single-particle Schrödinger equation for a free nonrelativistic particle.

To proceed on a slightly more formal level we expand the relativistic field operator (specializing to  $D = 2$ ) in a Fock basis,

$$\phi(t, x) = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_k}} [a_k e^{-i\omega_k t + ikx} + a_k^\dagger e^{i\omega_k t - ikx}], \tag{79}$$

with  $w_k \equiv \sqrt{k^2 + m^2}$  and the Fourier modes obeying  $[a_q, a_k^\dagger] = 2\pi \delta(k - q)$ . The field operator  $\phi$  changes particle number by one unit, and a one particle state  $|k\rangle$  with momentum  $k$  is defined by

$$|k\rangle = \sqrt{2\omega_k} a_k^\dagger |0\rangle \Rightarrow \langle q|k\rangle = 4\pi \omega_k \delta(q - k). \tag{80}$$

Hence, it makes sense to consider the truncated fields

$$\phi_{01}(t, x) \equiv \mathbb{P}_0 \phi(t, x) \mathbb{P}_1, \tag{81}$$

$$\phi_{10}(t, x) \equiv \mathbb{P}_1 \phi(t, x) \mathbb{P}_0, \tag{82}$$

where we have introduced the projections onto vacuum and one-particle sectors, respectively,

$$\mathbb{P}_0 \equiv |0\rangle\langle 0|, \quad \mathbb{P}_1 \equiv \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} |k\rangle \frac{1}{2\omega_k} \langle k|, \tag{83}$$

and we restrict the range of momentum to  $|k| \leq \Lambda \ll m$  in accordance with the nonrelativistic approximation. We then infer the  $t = 0$  commutator of the projected fields

$$[\phi_{01}(t, x), \phi_{10}(t, y)] = \mathbb{P}_0 \phi(t, x) \mathbb{P}_1 \phi(t, y) \mathbb{P}_0 - \mathbb{P}_1 \phi(t, y) \mathbb{P}_0 \phi(t, x) \mathbb{P}_1, \tag{84}$$

with the first (second) term obviously acting in the vacuum (one-particle) sector. Projecting onto the former we find

$$\mathbb{P}_0 [\phi_{01}(t, x), \phi_{10}(t, y)] \mathbb{P}_0 = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \frac{1}{2\omega_k} e^{ik(x-y)} \mathbb{P}_0 \tag{85}$$

$$= \frac{1}{2m} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \frac{1}{\sqrt{1 + k^2/m^2}} e^{ik(x-y)} \mathbb{P}_0. \tag{86}$$

For  $|k| \leq \Lambda \ll m$  the square root is approximately unity, and rescaling the fields with  $\sqrt{2m}$  we find the following analogue of (18),

$$2m \mathbb{P}_0 [\phi_{01}(t, x), \phi_{10}(t, y)] \mathbb{P}_0 \simeq \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} e^{ik(x-y)} \mathbb{P}_0 = \frac{\sin \Lambda(x - y)}{\pi(x - y)} \mathbb{P}_0. \tag{87}$$

The right-hand side is the one dimensional regulated delta function (69) multiplying the projection  $\mathbb{P}_0$ . Including the rescaling we thus identify  $\sqrt{2m}\phi_{01} \sim \Phi$  and  $\sqrt{2m}\phi_{10} \sim \Phi^\dagger$ , recovering (72) in the vacuum sector of the nonrelativistic theory.

Note that the projection formalism above is quite reminiscent of the old Tamm–Dancoff idea of truncating in particle number [35, 36]. If we expand the projected fields (81) and (82), we obtain explicitly

$$\phi_{01}(t, x) = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \frac{1}{2\omega_k} e^{-i\omega_k t + ikx} |0\rangle \langle k| \tag{88}$$

$$\phi_{10}(t, x) = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \frac{1}{2\omega_k} e^{i\omega_k t - ikx} |k\rangle \langle 0|, \tag{89}$$

which corresponds to a cutoff in particle number,  $N \leq 1$ . In other words, one essentially projects onto negative (positive) frequencies or the annihilation (creation) parts of the field, replacing

$$a_k \rightarrow |0\rangle\langle k|, \quad a_k^\dagger \rightarrow |k\rangle\langle 0|, \quad (90)$$

in the Tamm–Dancoff spirit. However, it seems obvious that this can only be consistent in a nonrelativistic context where energies and momenta are small compared to particle masses; in a relativistically covariant theory any large boost will spoil these scale hierarchies as boosts, being dynamical Poincarè generators [37], neither conserve energy nor particle number. We will briefly come back to these issues in the next section.

### 3.4. Transition amplitudes

What can we say about the behaviour of quantum amplitudes in the relativistic limit? Such amplitudes are described in the Schrödinger picture by gluing state wave functionals onto the Schrödinger functional  $\langle \varphi_f | e^{-i\hat{H}T} | \varphi_i \rangle$  which generates time evolution (see appendices A and B),

$$\langle \Psi_2 | e^{-i\hat{H}T} | \Psi_1 \rangle = \int \mathcal{D}\varphi_f \mathcal{D}\varphi_i \Psi_2^*[\varphi_f] \langle \varphi_f | e^{-i\hat{H}T} | \varphi_i \rangle \Psi_1[\varphi_i]. \quad (91)$$

As described in the appendices, the Schrödinger functional is characterized by temporal boundary terms which, in second-order theories, depend on both the field and its derivative, reflecting the fact that Cauchy data are required to determine time evolution. In first-order theories the boundary terms depend on the field and its conjugate and do not contain time derivatives (which are not required as data). We have seen how the action changes from a second- to first-order theory in the nonrelativistic limit, so let us now turn to these boundary terms.

A typical relativistic boundary term for a real scalar field, imposing the Dirichlet condition  $\phi(T, \mathbf{x}) = \varphi(\mathbf{x})$  at time  $t = T$ , is

$$i \int d^d x \varphi(x) \dot{\phi}(T, x) + \frac{i}{2} \int d^d x \Lambda' \varphi^2(x), \quad (92)$$

where  $\phi(t, \mathbf{x})$  is a functional integration variable obeying the boundary condition  $\phi(T, \mathbf{x}) = 0$ ,  $\varphi(\mathbf{x})$  is the boundary field and  $\Lambda'$  is a regularization of  $\delta(0)$ . As first noted by Stueckelberg [38] (see also [39]) and discussed in detail by Symanzik [40], placing sources on the boundary leads to divergences in perturbation theory when the field and its ‘image charges’ (which impose the boundary conditions on propagators) are placed at the same point in time— $\Lambda'$  is hence a short time regulator, which may also be seen as a UV regulator of the field momentum propagator (see appendix B).

In terms of the nonrelativistic degrees of freedom the terms (92) may be written, including a rescaling of the boundary data,  $\varphi \rightarrow \varphi/\sqrt{2m}$ ,

$$\begin{aligned} & \int d^d x \varphi(\mathbf{x}) [e^{-imT} \Phi(T, \mathbf{x}) - e^{imT} \Phi^\dagger(T, \mathbf{x})] \\ & + \frac{i}{m} \int d^d x \varphi(\mathbf{x}) [e^{-imT} \partial_t \Phi(T, \mathbf{x}) + e^{imT} \partial_t \Phi(T, \mathbf{x})^\dagger] \\ & + \frac{i}{2m} \int d^d x \Lambda' \varphi^2(\mathbf{x}). \end{aligned} \quad (93)$$

Transforming to momentum space and imposing a cutoff, we see that the second line of (93) goes like  $E/m$  and is hence suppressed in the nonrelativistic limit. As discussed above, this is

consistent with expectations because boundary terms in first-order theories should not depend on derivatives of the fields. The third line of (93), in momentum space, goes like  $\Lambda'/m$ . Such terms are also absent in first-order theories and we see that if we maintain  $m$  as the largest scale in our theory, this term is also suppressed in the nonrelativistic limit.

We are therefore left with the first line of (93), depending on the fields and not their momenta, with the condition

$$\phi(T, \mathbf{x}) = 0 \rightarrow e^{-imT} \Phi(T, \mathbf{x}) + e^{imT} \Phi^\dagger(T, \mathbf{x}) = 0, \tag{94}$$

up to small corrections, implying a mixed boundary condition on the nonrelativistic fields. We will discuss further limits of field theory wave functionals towards the end of the next section.

#### 4. Light-front quantization

In 1949 Dirac pointed out that, in a relativistically covariant theory, there are several alternative ‘forms of relativistic dynamics’ [37]. In particular, one may postulate field commutators on null planes rather than entirely space-like hypersurfaces leading to light-cone or, somewhat more precisely, light-front quantum field theory. The literature on this subject is vast and we refer the reader to the reviews [41–44] and the references cited therein.

One of the hopes of studying light-front quantum field theory was indeed that the Tamm–Dancoff approximation of the previous section might become feasible [45, 46] in a relativistic context. This hope was based on the fact that, upon quantizing on null-planes, a number of nonrelativistic features seem to arise within a fully relativistic approach. This was first noted by Weinberg in his analysis of the infinite momentum limit of Feynman graphs [47] (see also [48]) and can be made explicit in terms of a  $2d$  Galilei subgroup of the Poincaré group [49, 50]. Among the consequences one finds, for example, a separation of relative and centre-of-mass motion within bound states. Most interesting seems to be the closely related suppression of vacuum fluctuation and pair production effects expressed as the folkloric statement that the light-front vacuum is ‘trivial’ [42–44]. In what follows we will take a fresh look at the nonrelativistic aspects of light-front field theory in terms of spectral flow.

##### 4.1. Time-slice geometry

Field quantization on an arbitrary hypersurface (with time-like or light-like normals) may be formulated as follows [44]. We introduce a coordinate transformation  $x \rightarrow \xi$  (and likewise for momenta,  $p \rightarrow k$ ),

$$\xi^\alpha = L_\mu^\alpha x^\mu, \quad k_\alpha = L_\alpha^\mu p_\mu, \quad L_\mu^\alpha L_\beta^\mu = \delta_\beta^\alpha. \tag{95}$$

The new variables describe an alternative (3+1)-foliation of Minkowski space with  $\xi^0$  being the new time variable, conjugate to the momentum component  $k_0$ . We assume that the transformation is linear<sup>1</sup> so that

$$\Sigma : \xi^0 = \text{const} \tag{96}$$

is a hyperplane of equal time  $\xi^0$ . The metric associated with the transformation (95) is

$$g_{\alpha\beta} = \eta_{\mu\nu} L_\alpha^\mu L_\beta^\nu \equiv \begin{pmatrix} g_{00} & \mathbf{g}^T \\ \mathbf{g} & -G \end{pmatrix}, \tag{97}$$

<sup>1</sup> Clearly, this is not the most general case. Even in special relativity one can choose hyperboloids rather than planes as surfaces of equal time which corresponds to Dirac’s ‘point-form’ of relativistic dynamics [37].



where we have introduced a (3+1)-split in the last step. Hence,  $-G$  is the induced metric on the quantization hyperplane. The (constant) normal on  $\Sigma$  is

$$N_\mu = \left. \frac{\partial \xi^0}{\partial x^\mu} \right|_\Sigma = L_\mu^0, \quad (98)$$

and prominently enters the inverse metric which we write as follows:

$$g^{\alpha\beta} = \begin{pmatrix} N^2 & \gamma^T \\ \gamma & -\Gamma \end{pmatrix}. \quad (99)$$

The square of the normal can be expressed in terms of metric determinants from (97),

$$N^2 = g^{00} = \det(-G)/\det(g), \quad (100)$$

and will become important in a moment. The inverse metric (99) governs the mass-shell constraint,

$$0 = p^2 - m^2 = g^{\alpha\beta} k_\alpha k_\beta - m^2, \quad (101)$$

which will be used to determine the energy variable  $k_0$  in terms of the ‘spatial’ components  $k_a$ ,  $a = 1, 2, 3$ . We mention in passing that introducing the spacetime foliation  $(\xi^0, \xi^a)$  can be viewed as gauge-fixing the time reparametrization invariance  $\tau \rightarrow \tau'$  generated by the constraint (101). The associated Faddeev–Popov (FP) expression is [44]

$$\text{FP} \equiv N \cdot p \equiv N^2 k_0 + \gamma^a k_a, \quad (102)$$

which has to be evaluated on mass-shell, i.e. by expressing  $p_0$  and  $k_0$  in terms of  $p_i$  and  $k_a$ , respectively via (101). Expanding the latter we find the quadric

$$0 = N^2 k_0^2 + 2\gamma^a k_a k_0 - \Gamma^{ab} k_a k_b - m^2 = 0. \quad (103)$$

Interestingly, its discriminant  $\Delta$  basically coincides with the FP expression squared,

$$\Delta \equiv 4\{(\gamma^a k_a)^2 + N^2(\Gamma^{ab} k_a k_b + m^2)\} = 4\text{FP}^2. \quad (104)$$

Depending on the value of  $N^2$ , we thus have to consider two different cases. The generic one is that the normal  $N$  on  $\Sigma$  is *time-like*,  $N^2 > 0$ . In this case, the mass-shell constraint is of *second order* in  $k_0$ , so that there are two distinct solutions,

$$k_0 = \frac{1}{N^2}(-\gamma^a k_a \pm \sqrt{\Delta}/2) = \frac{1}{N^2}(-\gamma^a k_a \pm |\text{FP}|). \quad (105)$$

The second case to be considered is in a sense degenerate. It corresponds to a *light-like* normal,  $N^2 = 0$ . In this case, the constraint (103) is only of *first order* in  $k_0$  leading to a single solution,

$$k_0 = \frac{\Gamma^{ab} k_a k_b + m^2}{2|\text{FP}|}, \quad (106)$$

with  $\text{FP} = \gamma^a k_a$  according to (102). While (106) looks simpler than (105) (unique sign, no square root) a new difficulty arises as the momentum projection  $\text{FP} = \gamma^a k_a$  may be vanishing. We thus have a Gribov problem which will turn into the notorious zero mode problem of light-front field theory.

In what follows we want to study the light-like limit (LLL),  $N^2 \rightarrow 0$ .

#### 4.2. The LLL metric

To the best of our knowledge, a limiting approach to light-front coordinates was first suggested by Chen in 1971 [51]. His new coordinates differed from light-front ones by an infinitesimal rotation. Finite rotations were later considered in [52, 53].

Clearly, rotations preserve the orthogonality of the coordinates. For relativistic systems this is not a crucial issue, however, and one may as well give up orthogonality. This approach was first adopted by Prokhvatilov and Franke [54] and independently by Lenz *et al* [55]. Since then it has frequently been utilized for field theory applications both at zero and finite temperature (see, e.g., [56–58] and [59, 60], respectively). It has also been adopted for the matrix model approach to M-theory [61] where the notion of the ‘LLL’ was coined. The idea is to introduce the new coordinates

$$\begin{aligned} \xi^0 &= x^0(1 + \eta^2/2) + x^3(1 - \eta^2/2), \\ \xi^3 &= x^- \equiv x^0 - x^3, \\ \xi^{1,2} &= x^{1,2}, \end{aligned} \tag{107}$$

such that, in the limit  $\eta \rightarrow 0$ ,  $\xi^0$  becomes the standard light-front time,

$$x^+ = x^0 + x^3. \tag{108}$$

The invariant distance element is

$$ds^2 = d\xi^0 d\xi^3 - d\xi_{\perp}^2 - \frac{\eta^2}{2} d\xi^3 d\xi^3, \tag{109}$$

implying the following metric and its inverse

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1/2 & 0 & 0 & -\eta^2/2 \end{pmatrix}, \quad g^{\alpha\beta} = \begin{pmatrix} 2\eta^2 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \tag{110}$$

which we will henceforth refer to as the LLL metric. Comparing with (99) we read off that the hyperplane normal  $N$  on  $\Sigma : \xi^0 = 0$  satisfies

$$N^2 = 2\eta^2 > 0, \tag{111}$$

implying that  $N$  is time-like for  $\eta \neq 0$ . In accordance with that, the line element on  $\Sigma$  is space-like even for vanishing transverse separation,

$$ds^2|_{\xi^0=0=\xi_{\perp}} = -\frac{\eta^2}{2} d\xi^3 d\xi^3 < 0. \tag{112}$$

Hence, for  $\eta \neq 0$ ,  $\Sigma$  is indeed a space-like hyperplane. Figure 1 shows the mass shell energy–momentum relation (101) in our coordinates (107) and in light-front coordinates. The on-shell energies at  $\eta \neq 0$  are

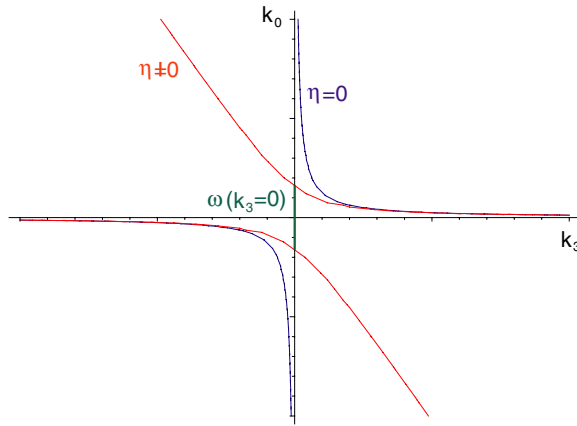
$$\omega^{\pm} = -\frac{k_3}{\eta^2} \pm \sqrt{\frac{k_3^2}{\eta^4} + \frac{k_{\perp}^2 + m^2}{2\eta^2}}. \tag{113}$$

The difference between these two energies will be denoted  $\omega$ ,

$$\omega := \omega^+ - \omega^- = 2\sqrt{\frac{k_3^2}{\eta^4} + \frac{k_{\perp}^2 + m^2}{2\eta^2}}, \tag{114}$$

where  $\omega(k_3 = 0)$  is the size of the gap in the energy spectrum of figure 1. Expanding the on-shell energies,

$$\omega^{\pm} = \begin{cases} \frac{k_{\perp}^2 + m^2}{4k_3} + \mathcal{O}(\eta^2) & \text{if } k_3 \gtrsim 0 \\ \pm \frac{1}{\eta} \sqrt{\frac{k_{\perp}^2 + m^2}{2}} & \text{if } k_3 = 0, \\ \frac{-2k_3}{\eta^2} - \frac{k_{\perp}^2 + m^2}{4k_3} + \mathcal{O}(\eta^2) & \text{if } k_3 \lesssim 0 \end{cases} \tag{115}$$



**Figure 1.** Mass shells for  $\eta \neq 0$  and in the LLL ( $\eta = 0$ ) where the signs of  $k_0 = p_+$  and  $k_3 = p_-$  coincide. Also displayed is the energy gap  $\omega(k_3 = 0)$ .  
(This figure is in colour only in the electronic version)

and referring to the four quadrants of figure 1 enumerated anticlockwise from the upper right, we see that the energies in the first and third quadrants remain finite as  $\eta \rightarrow 0$  and become the expected light-front energies, while those in the second and fourth quadrants flow to infinity. In the following subsection we will demonstrate explicitly how this spectral flow gives rise to noncommutativity in the fields.

4.3. Poisson brackets of the field

A scalar field obeying the LLL coordinate equations of motion may be written

$$\begin{aligned} \phi(\xi^0, \xi^a) &= \int \frac{dk_0 d^3k}{(2\pi)^4} e^{-ik_\mu \xi^\mu} \tilde{\phi}(k_0, k_a) 2\pi \delta \left[ \eta^2 k_0^2 + 2k_0 k_3 - \frac{1}{2}(k_\perp^2 + m^2) \right] \\ &= \int \frac{d^3k}{(2\pi)^3} e^{-i\omega^+ \xi^0 - ik_a \xi^a} \frac{\tilde{\phi}(\omega^+, k_a)}{\eta^2 \omega} + e^{-i\omega^- \xi^0 - ik_a \xi^a} \frac{\tilde{\phi}(\omega^-, k_a)}{\eta^2 \omega}. \end{aligned} \tag{116}$$

Here  $\omega^\pm$  are the on shell energies (113). The conjugate field momentum is given by

$$\begin{aligned} \Pi &= \eta^2 \partial_0 \phi + \partial_3 \phi \\ &= \int \frac{d^3k}{(2\pi)^3} -\frac{i}{2} e^{-i\omega^+ \xi^0 - ik_a \xi^a} \tilde{\phi}(\omega^+, k_a) + \frac{i}{2} e^{-i\omega^- \xi^0 - ik_a \xi^a} \tilde{\phi}(\omega^-, k_a), \end{aligned} \tag{117}$$

where the first line shows that the momentum contains the velocity  $\partial_0 \phi$  only as long as  $\eta \neq 0$ . For  $\eta = 0$ , however,  $\Pi$  is merely an abbreviation for the spatial derivative,  $\partial_3 \phi$ . It is easily verified that the equal time Poisson brackets

$$[\phi(\xi^0, \xi^a), \Pi(\xi^0, \xi'^a)] = i\delta^3(\xi^a - \xi'^a), \tag{118}$$

$$[\phi(\xi^0, \xi^a), \phi(\xi^0, \xi'^a)] = 0, \tag{119}$$

are equivalent to

$$[\tilde{\phi}(\omega^+, k_a), \tilde{\phi}(\omega^-, k'_a)] = (2\pi)^3 \eta^2 \omega \delta^3(k_a + k'_a), \tag{120}$$

with all other brackets vanishing. Referring again to the four quadrants of figure 1, we see that the bracket is non-zero only when one momentum space field has support in the first (second) quadrant and one in the third (fourth). It is precisely the cancellation between these two pairs of sectors which makes the Poisson bracket (119) of the field with itself vanish. This may be verified by splitting the field into two parts, defined by the range of integration over  $k_3$ , as so,

$$\begin{aligned} \phi(\xi^0, \xi^a) &= \int \frac{d^2 k_\perp}{(2\pi)^2} \int_0^\infty \frac{dk_3}{2\pi} e^{-i\omega^+ \xi^0 - ik_a \xi^a} \frac{\tilde{\phi}(\omega^+, k_a)}{\eta^2 \omega} + \int_{-\infty}^0 \frac{dk_3}{2\pi} e^{-i\omega^- \xi^0 - ik_a \xi^a} \frac{\tilde{\phi}(\omega^-, k_a)}{\eta^2 \omega} \\ &+ \int \frac{d^2 k_\perp}{(2\pi)^2} \int_{-\infty}^0 \frac{dk_3}{2\pi} e^{-i\omega^+ \xi^0 - ik_a \xi^a} \frac{\tilde{\phi}(\omega^+, k_a)}{\eta^2 \omega} + \int_0^\infty \frac{dk_3}{2\pi} e^{-i\omega^- \xi^0 - ik_a \xi^a} \frac{\tilde{\phi}(\omega^-, k_a)}{\eta^2 \omega}. \end{aligned} \tag{121}$$

These terms live in quadrants 1, 3, 2 and 4, respectively. The contribution from quadrants one and three to the commutator  $[\phi, \phi]$  is

$$\int \frac{d^3 k_a}{(2\pi)^3} \text{Sign}(k_3) e^{-ik_3(\xi - \xi')^3 - ik_\perp \cdot (\xi - \xi')^\perp} \frac{1}{\eta^2 \omega}, \tag{122}$$

while quadrants 2 and 4 contribute

$$- \int \frac{d^3 k_a}{(2\pi)^3} \text{Sign}(k_3) e^{-ik_3(\xi - \xi')^3 - ik_\perp \cdot (\xi - \xi')^\perp} \frac{1}{\eta^2 \omega}, \tag{123}$$

which together cancel to give (119). Thus, the commutativity of the fields on hypersurfaces of equal time  $\xi^0$  (expressing relativistic causality) is actually resulting from a delicate ‘interplay’ of different field modes. We will see in a moment that this ‘interplay’ depends crucially on the flow parameter  $\eta$ .

So let us now take the limit of vanishing  $\eta$ . As  $\eta \rightarrow 0$  terms in  $\phi$  corresponding to quadrants one and three have well defined limits, as may be read off from (115). The other terms, however, contain rapidly oscillating complex exponentials since  $\omega^\pm \sim k_3 \eta^{-2}$  in these quadrants (and  $\omega^\pm \sim \eta^{-1}$  on the boundary  $k_3 = 0$ ) and will be suppressed by said oscillations. Observing that  $\eta^2 \omega$  is finite as  $\eta \rightarrow 0$ ,  $\eta^2 \omega \rightarrow 2|p_-|$ , we are left with a truncated field,

$$\phi(x^+, x^a) = \int \frac{d^2 p_\perp}{(2\pi)^2} \int_0^\infty \frac{dp_-}{2\pi} e^{-iEx^+ - ip_a x^a} \frac{\tilde{\phi}(E, p_a)}{2|p_-|} + \int_{-\infty}^0 \frac{dp_-}{2\pi} e^{-iEx^+ - ip_a x^a} \frac{\tilde{\phi}(E, p_a)}{2|p_-|}, \tag{124}$$

where we have written

$$E \equiv E(p_-, p_\perp) = \frac{p_\perp^2 + m^2}{4p_-} = \frac{p_\perp^2 + m^2}{2p^+}. \tag{125}$$

The momentum space commutator is also well defined in this limit for  $p_- > 0$ ,  $p'_- < 0$ ,

$$[\tilde{\phi}(E, p_a), \tilde{\phi}(E, p'_a)] = 2|p_-| (2\pi)^3 \delta^3(p + p'), \tag{126}$$

and recalculating the Poisson bracket of the field with itself (this may also be read off from the  $\eta \rightarrow 0$  limit of (122)) we find

$$\begin{aligned} [\phi(x^+, x^a), \phi(x^+, x'^a)] &= \int \frac{d^2 p_\perp dp_-}{(2\pi)^3} \text{Sign}(p_-) e^{-ip_a(x-x')^a} \frac{1}{2|p_-|} \\ &= \delta^\perp(x - x') \int_{-\infty}^\infty \frac{dp_-}{2\pi} \frac{1}{2p_-} e^{-ip_-(x^- - x'^-)} \\ &= -\frac{i}{4} \delta^\perp(x - x') \text{Sign}(x^- - x'^-). \end{aligned} \tag{127}$$

Thus, we have finally arrived at the canonical Poisson bracket of the light-front field. Again, we see explicitly that spectral flow, causing the decoupling of high-energy states from the theory, alters the Poisson brackets, and therefore the commutators, of the theory. Following the flow all the way to the LLL one goes from a second- to a first-order theory, thereby inducing a noncommutativity in the configuration space of the original system.

It is useful to work in a mixed representation of light-front theory where the field depends on  $x^+$ ,  $p_-$  and  $x^\perp$  by defining, for  $p \equiv p_- > 0$ ,

$$\begin{aligned} \phi_p(x^+, x^\perp) &\equiv \sqrt{2|p|} \int \frac{d^2 p_\perp}{(2\pi)^2} e^{-iEx^+ - ip_\perp x^\perp} \tilde{\phi}(E, p_a), \\ \phi_p^\dagger(x^+, x^\perp) &\equiv \phi_{-p}(x^+, x^\perp). \end{aligned} \tag{128}$$

Using the commutation relations (126) it is easily checked that

$$[\phi_p(x^+, x^\perp), \phi_q^\dagger(x^+, x'^\perp)] = \delta^\perp(x - x') \delta(p - q). \tag{129}$$

So, for all  $x^+ \neq 0$  the limit  $\eta \rightarrow 0$  takes us, via spectral flow, to the light-front theory. Note though that our manipulations only hold for  $\xi^0 \neq 0$ , that is *off* the quantization hyperplane. Degrees of freedom are eliminated by rapid oscillations only for  $\xi^0 > 0$  but remnants survive at  $\xi^0 = 0$  ( $x^+ = 0$ ) in the limit. There are therefore extra degrees of freedom which remain in the quantization surface  $x^+ = 0$  and do not propagate into the bulk,  $x^+ > 0$ . It seems plausible that the boundary degrees of freedom are related to the notorious light-front zero modes (reviewed in [62]) as their propagator is instantaneous, namely proportional to  $\delta(x^+)$  [63] and hence indeed located at the temporal boundary. In the following subsections we will see that the same distinction between bulk and boundary arises in the functional picture.

#### 4.4. Light-front limit of wave functionals—the vacuum

As in previous sections, additional insight is provided by studying the behaviour of wave functionals in the light-front limit. We begin with the  $\eta^2 \neq 0$  vacuum wave functional  $\Psi_0[\varphi]$ , which may be written as a sum over all field histories beginning in the infinite past and intersecting the configuration  $\varphi(x)$  at time  $\xi^0 = 0$  (see [64–66] for applications in field theory, string theory and quantum gravity),

$$\Psi_0[\varphi] = \int \mathcal{D}\phi \exp - \int_{-\infty}^0 d\xi^0 L[\phi] \Big|_{\phi=\varphi \text{ at } \xi^0=0}, \tag{130}$$

where we have rotated to Euclidean space,  $\xi^0 \rightarrow -i\xi^0$ , and  $L[\phi]$  is the (free) Euclidean Lagrangian. The boundary condition at  $\xi^0 = -\infty$  is that the field should be regular.

The integral is computed by splitting  $\phi$  into a classical part which obeys the equation of motion and boundary conditions, and a quantum fluctuation which obeys a Dirichlet boundary condition at  $\xi^0 = 0$ . The general solution of the equations of motion is

$$\phi = \int \frac{d^3 k}{(2\pi)^3} e^{-ik_a \xi^a} e^{-\xi^0 \omega^+} \tilde{\phi}_1(k_a) + e^{-ik_a \xi^a} e^{-\xi^0 \omega^-} \tilde{\phi}_2(k_a), \tag{131}$$

where  $\omega^\pm$  are the on-shell energies of (113). The boundary conditions of (130) imply that  $\tilde{\phi}_1 = 0$  and  $\tilde{\phi}_2 = \tilde{\varphi}$ . The classical and quantum pieces are ‘orthogonal’ in that the action splits into two copies, one evaluated with the above solution and one evaluated with the quantum fields. In this way the integral may be performed to find

$$\Psi_0[\varphi] = Z(\eta) \exp \frac{1}{2} \int d^3 \xi d^3 \xi' \varphi(\xi^a) \Omega(\xi^a, \xi'^a) \varphi(\xi'^a) \tag{132}$$

with covariance

$$\Omega(\xi^a, \xi'^a) = \int \frac{d^3k}{(2\pi)^3} e^{-ik_a(\xi^a - \xi'^a)} \frac{\eta^2}{2} [\omega^-(k) - \omega^+(k)], \tag{133}$$

and where  $Z(\eta)$  is the contribution of quantum fluctuations. This may be determined from the normalization condition  $|\Psi_0|^2 = 1$ , which implies

$$Z^2(\eta) = \frac{1}{2} \int \mathcal{D}\varphi \exp -\frac{1}{2} \int d^3\xi d^3\xi' \varphi(\xi^a) [\Omega(\xi^a, \xi'^a) + \Omega^\dagger(\xi^a, \xi'^a)] \varphi(\xi'^a). \tag{134}$$

In the limit  $\eta \rightarrow 0$ , the product  $\eta^2 \omega^-(k_a)$  vanishes for all  $k_3 \geq 0$  and tends to  $-2k_3$  for  $k_3 < 0$ . We therefore find the LLL vacuum wave functional  $\Psi_0$ ,

$$Z(0) \exp - \int d^3x d^3x' \varphi(x) \varphi(x') \int \frac{d^2p_\perp}{(2\pi)^2} \int_0^\infty \frac{dp_-}{2\pi} e^{-ip_\perp(x^\perp - x'^\perp)} p_- e^{-ip_-(x^- - x'^-)}, \tag{135}$$

which may be simplified to give

$$\Psi_0[\varphi] = \text{Det}^{1/4}(i\partial_3) \exp - \int_0^\infty \frac{dp}{2\pi} \int d^2x^\perp \varphi_{-p}(x^\perp) p \varphi_p(x^\perp). \tag{136}$$

Here we have again transformed to the mixed representation with  $p \equiv p_-$  and  $\varphi_{-p} = \varphi_p^\dagger$ . The result (136) coincides with the light-front vacuum wave functional for free scalar fields found in [44]. Some remarks are in order at this point. First, one notes the interesting property that the covariance is local,

$$\Omega(p, p'; x^\perp, x'^\perp) = p \delta(p + p') \delta^2(x^\perp - x'^\perp), \tag{137}$$

unlike the original expression (133) for  $\eta \neq 0$ . Second, both positive and negative longitudinal momenta,  $\pm p \equiv \pm 2p^+$ , contribute in the exponent. Third, all mass dependence goes away in the LLL, when  $\eta^2 \omega^- \rightarrow -4p^+$ .

Thus, also from a functional viewpoint we see that the LLL, which we have described in terms of spectral flow ( $\eta \rightarrow 0$ ), has drastic effects on the Hilbert space of states. This will be corroborated in the final subsection below.

#### 4.5. Light-front limit of wave functionals—the Schrödinger functional

We now look at the  $\eta \rightarrow 0$  limit of the Schrödinger functional,

$$\langle \varphi_f | \exp -\hat{H}T | \varphi_i \rangle = \int \mathcal{D}\phi \exp - \int_0^T d\xi^0 L[\phi] \Big|_{\phi=\varphi^i \text{ at } \xi^0=0}^{\phi=\varphi^f \text{ at } \xi^0=T} \equiv \mathcal{N} \exp -S_E(\phi_{cl}). \tag{138}$$

Again the integral is performed by splitting the field into orthogonal quantum and classical pieces. Integrating over the quantum fluctuations yields the prefactor  $\mathcal{N}$ , which may be written as the inverse square root of the fluctuation determinant,  $\det \hat{\Delta}$ . Using the standard heat kernel identity

$$\mathcal{N} = -\frac{1}{2} \log \det \hat{\Delta} = -\frac{1}{2} \text{tr} \log \hat{\Delta} = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{tr} e^{-s\hat{\Delta}}, \tag{139}$$

a regulated determinant  $\mathcal{N}_a$  is defined by inserting a cutoff  $a$ ,

$$\mathcal{N}_a \equiv \int_a^\infty \frac{ds}{s} \text{tr} e^{-s\hat{\Delta}}. \tag{140}$$

The classical contribution follows from solving the classical boundary value problem

$$-\partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \phi - \sqrt{g} m^2 \phi = 0, \quad \phi(T, \mathbf{x}) = \varphi^f(\mathbf{x}), \quad \phi(0, \mathbf{x}) = \varphi^i(\mathbf{x}). \quad (141)$$

The general solution (131) now obeys

$$\tilde{\varphi}^i = \tilde{\varphi}_1 + \tilde{\varphi}_2, \quad \tilde{\varphi}^f = e^{-T\omega^+} \tilde{\varphi}_1 + e^{-T\omega^-} \tilde{\varphi}_2, \quad (142)$$

and it is straightforward to calculate the corresponding action. We find, schematically,

$$-S_E(\phi_{cl}) = \frac{1}{2} \varphi^f * A * \varphi^f + \frac{1}{2} \varphi^i * A * \varphi^i - \varphi^f * B * \varphi^i, \quad (143)$$

where the asterisks denote convolution integrals. The fields in the mixed representation depend on  $\xi^\perp$  and  $p \equiv k_3 \equiv p_-$  and the integral is over all  $p$  and  $\xi_\perp$ . The kernels are given by

$$\begin{aligned} A_p(\xi^\perp, \xi'^\perp) &= \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{\eta^2}{2} (\omega^- - \omega^+) \left[ \frac{1}{e^{T(\omega^+ - \omega^-)} - 1} + \frac{1}{1 - e^{T(\omega^- - \omega^+)}} \right] e^{-ik_\perp(\xi - \xi')^\perp}, \\ B_p(\xi^\perp, \xi'^\perp) &= \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{\eta^2}{2} (\omega^- - \omega^+) \left[ \frac{e^{-T\omega^-}}{e^{T(\omega^+ - \omega^-)} - 1} + \frac{e^{-T\omega^+}}{1 - e^{T(\omega^- - \omega^+)}} \right] e^{-ik_\perp(\xi - \xi')^\perp}. \end{aligned} \quad (144)$$

As  $\eta \rightarrow 0$  we find

$$\begin{aligned} A_p(\xi^\perp, \xi'^\perp) &\rightarrow |p| \delta^\perp(\xi - \xi'), \\ B_p(\xi^\perp, \xi'^\perp) &\rightarrow -p \theta(p) \int \frac{d^2 k}{(2\pi)^2} e^{-TE(p, k_\perp)} e^{-ik_\perp(\xi - \xi')^\perp}, \end{aligned} \quad (145)$$

where  $E(p, k_\perp) = (k_\perp^2 + m^2)/4p$  is the light-front energy. The Schrödinger functional (138) therefore becomes, in this limit,

$$\begin{aligned} \mathcal{N}_a \exp \left\{ - \int_0^\infty \frac{dp}{2\pi} \int d^2 \xi \varphi_{-p}^f(\xi^\perp) p \varphi_p^f(\xi^\perp) - \int_0^\infty \frac{dp}{2\pi} \int d^2 \xi \varphi_{-p}^i(\xi^\perp) p \varphi_p^i(\xi^\perp) \right. \\ \left. + \int_0^\infty \frac{dp}{2\pi} \int d^2 \xi d^2 \xi' \varphi_{-p}^f(\xi^\perp) \left[ \int \frac{d^2 k}{(2\pi)^2} p e^{-TE(p, k_\perp)} e^{-ik_\perp(\xi - \xi')^\perp} \right] \varphi_p^i(\xi') \right\}. \end{aligned} \quad (146)$$

Comparing with (136) the terms in the first line are readily identified with light-front vacuum wave functionals depending on only initial or final fields  $\varphi^i$  and  $\varphi^f$ , respectively. Hence, with the fields being stuck at  $x^+ = 0$  and  $x^+ = T$ , these terms are *non-propagating*. The final,  $T$ -dependent, term, on the other hand, *does* correspond to propagation being precisely the expression for the anti-holomorphic light-front transition amplitude of appendix A. We may thus write the LLL of the Schrödinger functional (138) in the compact form

$$\langle \varphi^f | e^{-H(\eta)T} | \varphi^i \rangle \rightarrow \mathcal{N}_a \Psi_0[\varphi^f] \langle \varphi^{f\dagger} | e^{-H(0)T} | \varphi^i \rangle \Psi_0[\varphi^i]. \quad (147)$$

Interestingly, one finds a phenomenon that might be called bulk-boundary decoupling: the total transition amplitude decomposes into a product of a bulk and two boundary pieces with the former to be identified with the first-order, light-front transition amplitude proper,  $\langle \varphi^{f\dagger} | e^{-H(0)T} | \varphi^i \rangle$ . This is consistent with the observation that only *half* of the original ( $\eta \neq 0$ ) degrees of freedom survive outside of the quantization planes as is manifest in that the bulk term contains the propagating modes  $\varphi_p^{f\dagger}$  and  $\varphi_p^i$  with only positive longitudinal momentum,  $p > 0$ . In Fock space language these modes correspond to creation and annihilation terms corroborating the interpretation that the propagating  $\varphi$ 's have indeed become light-front fields.

The bulk-boundary decoupling, with surface modes  $\varphi_{-p}^f$  and  $\varphi_p^i$ , for  $p$  both positive and negative, attached to the quantization hypersurfaces, echoes the result of section 4.3 where we saw that the LLL correctly reproduces the light-front commutation relations only *off* the quantization surface, i.e., in the bulk.

## 5. Discussion and conclusions

We have discussed limits of several quantum systems in which second-order terms in the action are suppressed. The most striking feature of these limits is that noncommutativity of configuration space appears as an *emergent phenomenon* resulting from a unifying principle, namely *spectral flow*. The limits in question may then be described in terms of a generic flow parameter  $\lambda$  with  $\lambda \rightarrow 0$ .

We have seen through various applications that it is the consequential truncation of state space which leads to features of the first-order theory which one expects from a naive treatment of the classical action. For example, the change of conjugate momenta to fields rather than their derivatives appears through incomplete cancellations of mode commutators due to missing states, and the expected preservation of particle number in nonrelativistic field theory appears as a restriction on the allowed interactions controlled by the size of the available energy momentum space.

In the functional picture we have examined the Schrödinger and vacuum functionals. In a Schrödinger representation it is only these two objects which are required to build correlation functions and  $S$ -matrix elements in perturbation theory. We remark that the form of the boundary terms in the Schrödinger functional, as described by Symanzik [40], are key to understanding renormalization in the Hamiltonian formalism. This opens up the possibility to interpret the spectral flow presented here as a *renormalization group (RG) flow* with the noncommutativity limit  $\lambda \rightarrow 0$  corresponding to special RG fixed points. If this is feasible, the difficult renormalization problem of light-front field theory, for instance, might be attacked from this new vantage point. It is our intention to address this issue in a future publication.

## Acknowledgments

The authors are grateful to Andreas Wipf for very useful discussions and AI thanks Matthew Daws for helpful correspondence.

## Appendix A. Transition amplitudes in the anti-holomorphic representation

Given a theory with commutator  $[\hat{\phi}(t, \mathbf{x}), \hat{\phi}^\dagger(t, \mathbf{x}')] = \delta^d(\mathbf{x} - \mathbf{x}')$ , where  $\mathbf{x}$  is shorthand for any set of  $d$  dependent variables, and a Hamiltonian  $\hat{H}(\phi^\dagger, \phi)$  one may describe the abstract space of states  $|\Psi, t\rangle$  by wave functionals  $\Psi[\varphi^\dagger, t] \equiv \langle \varphi^\dagger | \Psi, t \rangle$  of a complex conjugate field, on which the operation of  $\hat{\phi}^\dagger$  is multiplicative and  $\hat{\phi}$  acts as a derivative,

$$\hat{\phi}^\dagger(0, \mathbf{x}) \rightarrow \varphi^\dagger(\mathbf{x}), \quad \hat{\phi}(0, \mathbf{x}) \rightarrow \frac{\delta}{\delta \varphi^\dagger(\mathbf{x})}. \quad (\text{A.1})$$

In this representation the time dependence of physical states is controlled by the Schrödinger equation,

$$i \frac{\partial}{\partial t} \Psi[\varphi^\dagger, t] = \hat{H}\left(\varphi^\dagger, \frac{\delta}{\delta \varphi^\dagger}\right) \Psi[\varphi^\dagger, t], \quad (\text{A.2})$$

which may be exponentiated to give

$$\Psi[\varphi^\dagger, t] = \int \mathcal{D}\varphi'^\dagger \mathcal{D}\varphi' \langle \varphi^\dagger | e^{-i\hat{H}t} | \varphi' \rangle \Psi[\varphi'^\dagger, 0]. \quad (\text{A.3})$$



The Schrödinger functional,  $\langle \varphi^\dagger | e^{-i\hat{H}t} | \varphi' \rangle$ , may be described by a functional integral following the usual procedure of discretizing the time interval and inserting complete sets, which are given in this representation by

$$1 = \int \mathcal{D}\varphi^\dagger \mathcal{D}\varphi |\varphi\rangle \exp \left\{ - \int d^d x \varphi^\dagger(\mathbf{x}) \varphi(\mathbf{x}) \right\} \langle \varphi^\dagger |. \quad (\text{A.4})$$

Note that the measure in this expression is over fields on a constant time hypersurface. The resulting functional integral is

$$\begin{aligned} \langle \varphi^\dagger | e^{-i\hat{H}T} | \varphi' \rangle = & \int \mathcal{D}\varphi^\dagger \mathcal{D}\varphi \exp \left[ \frac{1}{2} \int d^d x \varphi^\dagger(\mathbf{x}) \phi(t, \mathbf{x}) + \frac{1}{2} \int d^d x \phi^\dagger(0, \mathbf{x}) \varphi'(\mathbf{x}) \right. \\ & \left. + i \int_0^T dt \int d^d x \frac{1}{2i} (\dot{\phi}^\dagger \phi - \phi^\dagger \dot{\phi}) - \mathcal{H}(\phi^\dagger, \phi) \right] \Bigg|_{\phi(0, \mathbf{x}) = \varphi'(\mathbf{x})}^{\phi^\dagger(T, \mathbf{x}) = \varphi^\dagger(\mathbf{x})}, \end{aligned} \quad (\text{A.5})$$

where  $\mathcal{H}$  is the Hamiltonian density. This is the transition amplitude for first-order theories derived in the anti-holomorphic representation using coherent states [67]. An equivalent expression, which collects all dependence on the boundary fields into boundary terms in the action, is

$$\begin{aligned} \langle \varphi^\dagger | e^{-i\hat{H}T} | \varphi' \rangle = & \int \mathcal{D}\varphi^\dagger \mathcal{D}\varphi \exp \left[ \int d^d x \varphi^\dagger(\mathbf{x}) \phi(T, \mathbf{x}) + \int d^d x \phi^\dagger(0, \mathbf{x}) \varphi'(\mathbf{x}) \right. \\ & \left. + i \int_0^T dt \int d^d x \frac{1}{2i} (\dot{\phi}^\dagger \phi - \phi^\dagger \dot{\phi}) - \mathcal{H}(\phi^\dagger, \phi) \right] \Bigg|_{\phi(0, \mathbf{x}) = 0}^{\phi^\dagger(t, \mathbf{x}) = 0}. \end{aligned} \quad (\text{A.6})$$

This may be derived either by a rearrangement of terms in the discretized product, or in the continuum limit using the change of variables

$$\phi^\dagger(t, \mathbf{x}) \rightarrow \phi^\dagger(t, \mathbf{x}) + \theta(t - T) \varphi^\dagger(\mathbf{x}), \quad \phi \rightarrow \phi(t, \mathbf{x}) + \theta(-t) \varphi'(\mathbf{x}), \quad (\text{A.7})$$

where the new variables obey  $\phi^\dagger(T, \mathbf{x}) = \phi(0, \mathbf{x}) = 0$ . This change of variables corresponds to a separation of degrees of freedom on the boundary, which are fixed by boundary conditions, and in the bulk, which are integrated over.

We will finally give the explicit form of the Schrödinger functional for the (Euclidean) light-front field theory of section 4 where we have commutation relations as in (129),

$$[\phi_p(x^+, x^\perp), \phi_q^\dagger(x^+, x'^\perp)] = \delta^\perp(x - x') \delta(p - q), \quad (\text{A.8})$$

and Hamiltonian density

$$\mathcal{H} = \frac{1}{4} \phi_p^\dagger(x^+, x^\perp) (\Delta_\perp^2 - m^2) \phi_p(x^+, x^\perp). \quad (\text{A.9})$$

The integral in (A.5) is evaluated by first identifying the solution of the classical equations of motion, which follow from (A.8) and (A.9), subject to the boundary conditions  $\phi_p^\dagger(T, x^\perp) = \varphi_p^\dagger(x^\perp)$ ,  $\phi_p(0, x^\perp) = \varphi'_p(x^\perp)$ . The integration variable may be decomposed into this field and an orthogonal quantum fluctuation, the integral over the latter giving a determinant factor which stands in need of regularization. One finds

$$\langle \varphi^\dagger | e^{-\hat{H}T} | \varphi' \rangle = \text{const.} \exp \int_0^\infty \frac{dp}{2\pi} \int d^2 x d^2 x' \varphi_p^\dagger(x^\perp) B_p(x^\perp, x'^\perp) \varphi'_p(x'^\perp) \quad (\text{A.10})$$

with kernel

$$B_p(x^\perp, x'^\perp) = \int \frac{d^2 k}{(2\pi)^2} p e^{-TE(p, k_\perp)} e^{-ik_\perp(x-x')^\perp}, \quad (\text{A.11})$$

and light-front energy  $E(p, k_\perp) = (k_\perp^2 + m^2)/4p$ . As discussed in section 4.5, one may view the light-front theory as the  $\eta \rightarrow 0$  limit of field theory in the LLL metric. We have seen that the functional (A.10) reappears in this limit as the time dependent (bulk) piece of the LLL Schrödinger functional at  $\eta = 0$ .

## Appendix B. Transition amplitudes in phase space

The representation given above is appropriate for theories with an action which is linear in time derivatives and is analogous to the phase space representation for theories with actions quadratic in time derivatives. Here, using a real scalar field to illustrate, it is common to represent states by wave functionals  $\Psi[\phi, t]$ . We have the algebra  $[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta^d(\mathbf{x} - \mathbf{y})$ , a Hamiltonian density  $\mathcal{H}(\phi, \pi)$  and complete sets

$$1 = \int \mathcal{D}\phi |\varphi\rangle \langle\varphi| = \int \mathcal{D}\pi |\pi\rangle \langle\pi|. \quad (\text{B.1})$$

The Schrödinger functional may be constructed by discretizing the time interval and inserting complete sets, where we find

$$\begin{aligned} \langle\varphi_f| e^{-i\hat{H}T} |\varphi_i\rangle &= \int \mathcal{D}\phi \mathcal{D}\pi \exp \left[ \frac{i}{2} \int d^d x \varphi_f(\mathbf{x}) \pi(t, \mathbf{x}) - \frac{i}{2} \int d^d x \pi(0, \mathbf{x}) \varphi_i(\mathbf{x}) \right. \\ &\quad \left. + i \int_0^T dt \int d^d x \frac{1}{2} (\pi \dot{\phi} - \dot{\pi} \phi) - \mathcal{H}(\phi, \pi) \right] \Bigg|_{\phi(0, \mathbf{x})=\varphi_i(\mathbf{x})}^{\phi(T, \mathbf{x})=\varphi_f(\mathbf{x})}. \end{aligned} \quad (\text{B.2})$$

Equivalently,

$$\begin{aligned} \langle\varphi_f| e^{-i\hat{H}T} |\varphi_i\rangle &= \int \mathcal{D}\phi \mathcal{D}\pi \exp \left[ i \int d^d x \varphi_f(\mathbf{x}) \pi(t, \mathbf{x}) - i \int d^d x \pi(0, \mathbf{x}) \varphi_i(\mathbf{x}) \right. \\ &\quad \left. + i \int_0^T dt \int d^d x \pi \dot{\phi} - \mathcal{H}(\phi, \pi) \right] \Bigg|_{\phi(0, \mathbf{x})=0}^{\phi(T, \mathbf{x})=0}. \end{aligned} \quad (\text{B.3})$$

When the Hamiltonian density is of the form  $\mathcal{H}(\phi, \pi) = \pi^2/2 + V(\phi)$  the momentum integration may be carried out to leave a configuration space integral over the exponent of the classical action,

$$\begin{aligned} \langle\varphi_f| e^{-i\hat{H}T} |\varphi_i\rangle &= \int \mathcal{D}\phi \exp \left[ i \int d^d x \varphi_f(\mathbf{x}) \dot{\phi}(t, \mathbf{x}) - i \int d^d x \varphi_i(\mathbf{x}) \dot{\phi}(0, \mathbf{x}) \right. \\ &\quad \left. + \frac{i}{2} \int d^d x \Lambda \varphi_f^2(\mathbf{x}) - \frac{i}{2} \int d^d x \Lambda \varphi_i^2(\mathbf{x}) + i \int_0^T dt \int d^d x \mathcal{L}(\phi(t, \mathbf{x})) \right] \Bigg|_{\phi(0, \mathbf{x})=0}^{\phi(T, \mathbf{x})=0}. \end{aligned} \quad (\text{B.4})$$

Here  $\Lambda$  is a regularization of  $\delta(0)$  which arises from the UV (short distance) divergent behaviour of the trivial field momentum propagator  $\delta(t - t')\delta^d(\mathbf{x} - \mathbf{x}')$ .

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